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Efficient Eigenvector Sensitivities by a New Procedure Based on Lanczos Vectors

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A new procedure, based on the use of Lanczos vectors, for the efficient computation of eigenvector sensitivities to changes in system parameters is presented. The method is based on a matrix reduction that uses the same Lanczos vectors as those used to obtain the original eigenvectors. Thus, the equation systems that are solved can be greatly reduced from the original matrix sizes. An explanation of why the method is most accurate for the lowest eigenvectors' derivatives is offered. Numerical results for two modest-sized examples are supplied, from which trends in the method's accuracy are suggested.

Nomenclature

$[A]$ = dynamic matrix defined by Eq. (2)
$[I]$ = identity matrix
$[K]$ = stiffness matrix
$[L]$ = lower decomposition of $[K]$
M = size of reduced system
$[M]$ = mass matrix
N = size of original system
$[T]$ = tridiagonal matrix
$[V]$ = matrix composed of Lanczos vectors
$\{X\}$ = N eigenvectors defined by Eq. (5)
$\{Y\}$ = M eigenvectors defined by Eq. (6)
λ = eigenvalues of system
$\{\phi_i\}$ = N eigenvectors defined by Eq. (4)
ω = natural frequencies of system

Subscripts

i, j = indices for typical eigenvectors

Superscripts

$'$ = partial differentiation with respect to a design parameter
T = matrix transpose
-1 = matrix inverse

Introduction

THE use of Lanczos vectors to efficiently and accurately reduce very large eigenvalue problems to a size that can be solved within core and that involves reasonable computer running times was first proposed some 25 years ago.[1] That procedure and slight variations of it are now used successfully on a regular basis throughout the world. Since then, the need to determine eigenvector sensitivity has become important in many design optimization and test/analysis correlation studies. However, the application of Lanczos vectors to this problem has not been as successful because of the need to judiciously select a Lanczos starting vector[2] to obtain reasonable sensitivity accuracy, and because computational running times have not been impressive when compared to competitive eigenvector sensitivity schemes.[3]

A new reduction procedure for determining eigenvector sensitivities is supplied. It makes use of the same Lanczos vectors as those used to obtain the original eigenvectors. As a result, there is no need to judiciously select the starting vector and the computational efficiency is greatly improved because there is no need to generate any additional Lanczos vectors (unless, as will be shown, greater accuracy is required for the upper half of the modal spectrum). In addition, it is demonstrated that the eigenvectors of the original system are identical to the eigenvectors of the coefficient matrix for their derivatives. Therefore, it is reasonable to expect that the same Lanczos vectors will be highly accurate for determining the system's eigenvector derivatives as well.

Furthermore, it is shown that the proposed method is more efficient than other eigenvector sensitivity schemes, such as Nelson's[4] and other methods.[5] The reason for this is that there is no need to solve $N \times N$ systems of simultaneous equations for the original N degree-of-freedom (DOF) systems, but only $M \times M$ systems of equations, where M represents the number of Lanczos vectors already generated, which is generally very much smaller than N.

Background

The newly proposed method starts with the modal vectors $\{\phi_i\}$ and frequencies ω_i obtained from a Lanczos-vector-based reduction method (see, e.g., Ojalvo[6]). As a result of this type of algorithm, the lower triangular matrix $[L]$ is available, where

$$[K] = [L][L]^T \tag{1}$$

as is the dynamic matrix $[A]$, where

$$[A] = [L]^{-1}[M][L]^{-T} \tag{2}$$

In addition, the mode shapes $\{\phi_i\}$ have been computed from

$$\{\phi_i\} = [L]^{-1}\{X_i\} \tag{3}$$

where

$$[K]\{\phi_i\} = \omega_i^2[M]\{\phi_i\}, \qquad \{\phi_i\}^T[M]\{\phi_i\} = 1/\omega_i^2 = \lambda_i \tag{4}$$

and

$$[A]\{X_i\} = \lambda_i\{X_i\}, \qquad \{X_i\}^T\{X_i\} = 1 \tag{5}$$

The $\{X_i\}$ are obtained after solution of the reduced $(M \times M)$ eigenvalue problem

$$[T]\{Y_i\} = \lambda_i\{Y_i\}, \qquad \{Y_i\}^T\{Y_i\} = 1 \tag{6}$$

has been solved within core, where $[T]$ of order M and is given by

$$[T] = [V]^T[A][V], \qquad [V]^T[V] = [I] \tag{7}$$

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and $[V]$ is the $N \times M$ matrix of M Lanczos vectors (which reduced the $N \times N$ dynamic matrix $[A]$).

The initial reduction process started with the approximation

$$\{X_i\} = [V]\{Y_i\} \tag{8}$$

where the columns of $[V]$ are the Lanczos vectors generated by the user judgment-free algorithm presented elsewhere,[6] or by some equivalent algorithm based on the use of Lanczos-type vectors.

Technical Approach

The newly proposed procedure employs an extension of the original approximation, as defined by Eq. (8),[1] and (a very special) Nelson-type approximation,[4] which is key to this newly proposed approach, i.e.,

$$\{\phi_i\}' = [L]^{-T}[V](\{z_i\} + c_i\{y_i\}) \tag{9}$$

where $\{z\}$ is an M-element vector, as is $\{Y\}$. After differentiation of Eq. (4) and some manipulation, this results in the $M \times M$ singular system of linear equations:

$$([T] - \lambda_i[I])\{z_i\} = -\lambda_i[V]^T[L]^{-1}\{b_i\} \tag{10}$$

in which

$$\{b_i\} = (\omega_i^2)'[M]\{\phi_i\} - ([K]' - \omega_i^2[M]')\{\phi_i\} \tag{11}$$

The solution is then completed after computation of the scaler c_i from the equation

$$c_i = -\left(\tfrac{1}{2}\right)(\omega_i^2)'/(\omega_i^2) - \omega_i^2\{Y_i\}^T[T]\{z_i\} - \left(\tfrac{1}{2}\right)(\omega_i^2)\{\phi_i\}^T[M]'\{\phi_i\} \tag{12}$$

The frequency derivative, required in Eqs. (11) and (12), is computed from the well-known equation

$$(\omega_i^2)' = (\omega_i^2)\{\phi_i\}^T([K]' - \omega_i^2[M]')\{\phi_i\} \tag{13}$$

The singular Eq. (10) may be solved efficiently using either Nelson's method,[4] epsilon decomposition,[7] or other known convenient methods.[3] Therefore, the entire process involves a straightforward computation using Eqs. (9–13), none of which requires solution of $N \times N$ systems and only one of which requires solution of an $M \times M$ system of equations, once the matrices and vectors already obtained in determining the mode shapes and frequencies have been retained and made available.

Eigenvectors of the Derivative Coefficient Matrix

It is well known that the Lanczos vector approximation for the $\{\phi_i\}$ are most accurate for the lower indexes i. Therefore, it is reasonable to inquire if the eigenvector derivative procedure proposed herein also will be most accurate for the lower indexed $\{\phi_i\}'$ as well. Some insight can be gained into the answer to this question by the following considerations.

Differentiation of the first of Eqs. (4) yields

$$([K] - \omega_i^2[M])\{\phi_i\}' = \{b_i\} \tag{14}$$

Thus, the coefficient matrix of $\{\phi_i\}'$ in Eq. (14) is identical to the coefficient matrix of the corresponding eigenvalue problem given by Eq. (4), once $\omega_i^2[M]\{\phi_i\}$ is subtracted from both sides of Eq. (4), i.e.,

$$([K] - \omega_i^2[M])\{\phi_j\} = (\omega_j^2 - \omega_i^2)[M]\{\phi_j\} \tag{15}$$

Therefore, the eigenvectors of Eqs. (15) and (4) are identical to the eigenvectors of the coefficient matrix in Eq. (14). Only their eigenvalues differ by the shift value ω_i^2.

Now, the lower eigenvectors of a matrix are generally excellent vectors for series-expansion approximation solutions involving that matrix. Thus, it is reasonable to expect that the Lanczos vectors, which give accurate approximations for the $\{\phi_i\}$, also will yield accurate approximations for the $\{\phi_i\}'$. However, the frequencies associated with each eigenvector of Eq. (15) have been shifted downward by ω_i^2 in Eqs. (14) and (15). Thus, the relative importance, or contribution, of each eigenvector in a modal expansion for $\{\phi_i\}'$ will be dependent not only on the right-hand side of Eq. (14), but on how small ω_i^2 is as well. Thus, the higher indexed $\{\phi_i\}'$ will generally require more expansion vectors with indices in the vicinity of i, for accuracy comparable to the derivatives for the lower values of i.

In addition, experience has shown that M Lanczos vectors, used to obtain a given number of accurate eigenvectors (i.e., approximately $\frac{1}{2}$–$\frac{2}{3}$ M), also contain valuable system information regarding the higher system modes. Thus, it is anticipated that the proposed Lanczos vector procedure for the eigenvector derivatives will be more accurate using M Lanczos vectors than $M/2$- or $2M/3$-accurate eigenvectors in an expansion approximation such as

$$\{\phi_i\}' = \sum_j a_j\{\phi_j\}, \qquad j = 1, 2, \ldots, \frac{M}{2} \quad \text{or} \quad \frac{M}{3} \tag{16}$$

where

$$a_j = \frac{(\{\phi_j\}^T\{b_i\})\omega_j^2}{(\omega_j^2 - \omega_i^2)}, \qquad j \neq i$$

and a_i is equal to c_i, as given by Eq. (12).

Comment on Repeated Eigenvalues

Although it is not the purpose of this paper to explore the accuracy of the present procedure for the case of repeated eigenvalues, the question of extending Eq. (9) to such cases has been raised more than once. In answer to this, it is proposed that a logical extension of the present approximate method for repeated eigenvalues would be to replace the $c_i\{Y_i\}$ term in Eq. (9) by the terms

$$\sum_{j=1}^{k} c_{ij}\{Y_{ij}\}$$

(similar to the manner first proposed by Ojalvo[8]), where the $\{Y_{ij}\}$ are the k eigenvectors corresponding to the k repeated eigenvalues ω_{ij} and the c_{ij} are obtained along the lines given by Mills-Currin[9] and others.

Numerical Applications

The proposed algorithm has been applied to two modest-sized numerical examples based on structural dynamic models involving 27 and 48 dynamic DOF, respectively. In the results presented, eigenvector (EV) and eigenvector derivative (EVD) accuracy sufficiency is defined as follows.

Let

$$\{a\} = \phi(\text{EV}) \quad \text{or} \quad \phi'(\text{EVD})$$

and

$$e = \frac{\{\Delta a\}^T\{\Delta a\}}{\{a\}^T\{a\}}, \qquad \{\Delta a\} = \{a\} - \{\bar{a}\}$$

where $\{a\}$ is the exact vector and $\bar{a}$ is its approximation.

Then $\{\bar{a}\}$ is considered sufficiently accurate if $e < 0.01$.

For the first example, a two-dimensional model termed FORK was selected (see Fig. 1). It consists of nine beam elements with distributed mass and extensional stiffness to which six concentrated masses, each with in-plane rotational inertias, are attached. Of the possible 27 (N) linearly independent Lanczos vectors that could be generated for this model, we have obtained EVs and EVDs using between 13 and 20 (M) Lanczos vectors. Table 1 indicates that these have yielded between 6 and 17 accurate EVs, but only between 2 and

Table 1 FORK model (N = 27 DOF)

M (No. of Lanczos vectors)	13	13	13	14	14	14	15	18	20
No. of accurate EVs	6	6	6	10	10	10	10	12	17
No. of accurate EVDs	2	3	4	5	6	7	8	9	10

Table 2 Two-story frame (N = 48 DOF)

M (No. of Lanczos vectors)	15	16	31	36	41	41
No. of accurate EVs	6	7	14	16	25	25
No. of accurate EVDs	3	6	7	8	9	10

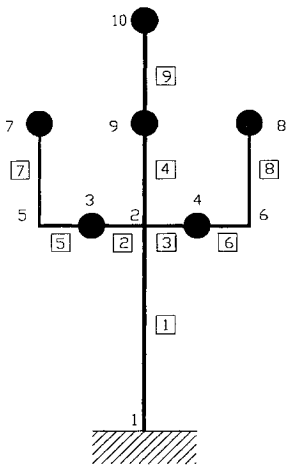

Fig. 1 Fork model geometry: 1, 2, . . . , nodal points; [1] . . . , elements; and ●, lumped masses with rotational inertias.

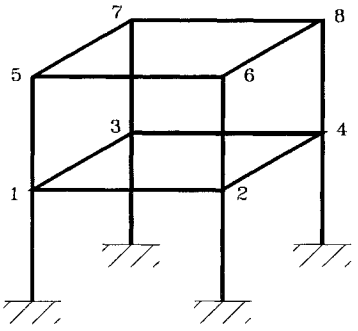

Fig. 2 Two-story frame: N = 48 DOF (6 DOF/ node—8 free nodes) and 1, 2 . . . , nodal points.

10 accurate EVDs. For the computations reported upon herein, the EVDs were obtained with respect to the in-plane rotational inertia of node 3 of Fig. 1.

The second example consists of a uniform 16-beam frame structure with 8 free nodes and 6 DOF per node (see Fig. 2). For this case, in which 48 (N) linearly independent Lanczos vectors were possible, we generated between 15 and 41 (M) such vectors. This yielded between 6 and 25 accurate EVs (see Table 2), but only between 3 and 10 accurate EVDs. For this example we obtained EVDs with respect to the cross-sectional area of one of the lower structural members.

Conclusions

A highly efficient algorithm for EVD determination has been proposed for large systems in which a Lanczos-vector-based approach has been used to determine the eigenvalues and eigenvectors. This efficiency has resulted from reuse of the same M Lanczos vectors to determine the EVDs, where M may be one or two orders of magnitude smaller than N (the size of the original system, which could contain thousands, or even tens of thousands, of DOF). Although EVD applications using the present new procedure have yet to be made to such large systems, it is well known that Lanczos-vector approaches are highly accurate and economical for computing the EVs of complex systems.

In the only two numerical studies performed to date and presented herein for determining the EVDs, it was found that the number of accurate EVDs was only approximately half the number of accurate EVs (when using the same Lanczos vectors). Hence, although we have come to anticipate that one-half to two-thirds of the M EVs obtained using M Lanczos vectors will be accurate, it is logical to anticipate that only one-quarter to one-third of the EVD vectors will be accurate. Nevertheless, if the preliminary trend continues for numerical studies involving over 1000 DOF (which are presently contemplated for the near future), the present method of EVD determination should prove highly valuable.

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